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Abstract

This article presents a simple model to determine the location strategies of two retail firms planning to open a number of stores in a geographical market. Firms try to maximize their profit under a leader-follower type competition in which the number of stores is made endogenous by the introduction of fixed costs. A novel methodology is developed, in which firms' strategies are defined in terms of heir location densities. This methodology leads to a model that is solvable analytically and to several results on competitive location strategies. First, it is shown that if the follower decides to enter a market, he enters at least as strongly as the leader. Second, the leader can effectively deter entry even if she is severely cost disadvantaged. However, in some cases the leader is better off by allowing the follower to enter the market. Third, the leader may also let the follower enter the market in some situations where she has a cost advantage. It is also shown that in situations where both firms enter the market, their location strategies are quite insensitive to model parameters.

Keywords: facilities, equipment planning, location, continuous.

Résumé

Cet article présente un modèle pour la détermination des stratégies de localisation de deux détaillants sur le point d'ouvrir des magasins dans un marché géographique. Les firmes essaient de maximiser leur profit dans un contexte de compétition premier/second joueur dans lequel le nombre de magasins est endogène à cause de l'introduction de coûts fixes. On développe une nouvelle méthodologie par laquelle les stratégies des firmes sont définies par leur densité de localisation. Cette méthodologie donne lieu à un modèle qui se résout de façon analytique et à plusieurs résultats sur la localisation compétitive. On montre d'abord que si le second joueur entre dans le marché, il doit le faire avec la même intensité que le premier joueur. Deuxièmement, dans certains cas, le premier joueur peut effectivement prévenir l'entrée du second joueur même si ses coûts fixes le désavantage. Toutefois, il existe des cas où le premier joueur a avantage à permettre l'entrée du second joueur dans des situations où sa structure de coûts le favorise. Finalement, on montre que dans les cas où les deux joueurs entrent dans le marché, leurs stratégies de localisation sont passablement insensibles aux paramètres du modèle.

Mots-clefs : installations, planification d'équipements, localisation continue.

1 Introduction

In November 2000, Starbucks, the world's leading gourmet coffee retailer, announced that it would open around 50 to 75 coffee shops in the Province of Quebec in the following four years. At the time, it had only five outlets compared with the 38 stores of Second Cup, the Canadian leader in the industry. Perhaps foreseeing the threat, Second Cup had earlier abandoned its expansion plans in the US and decided to concentrate its efforts on its Canadian business. Within fifteen months of the announcement, it opened 18 new shops, increasing its number to 56. During the same period Starbucks was able to open only three more shops and later opened another four, for a total of 12 by December 2003. If the expansion observed in these three years is any indication, it is fair to conclude that Starbucks will obtain less than a quarter of the market it had hoped to capture initially. Although there may be other reasons why Starbucks' plan did not materialize, it is quite plausible that Second Cup's expansion played a role.

The above story is not uncommon. The players, the nature of businesses, and the outcomes may differ in other contexts but the dynamics are very similar. One of the ways multi-store firms prevent or contain entry is through location decisions. For example, the success of 7-Eleven Japan has also been attributed to its expansion policy. When this firm enters a new market, it does so aggressively, by opening clusters of 50 to 60 stores at a time. Even though it has a limited presence geographically, in regions where it is present stores tend to be dense, which gives it not only distribution efficiency but also market dominance (Chopra 2000). As a result of its expansion policy, 7-Eleven Japan has increased its presence from 5,500 stores in only 21 of the 47 prefectures to more than 10,000 stores in 32 prefectures between 1994 and 2003.

Location decisions are of critical importance to all firms. Opening, closing, and relocating facilities require extensive planning due to the strategic nature of these decisions. When facilities do not have physical contact with customers (such as plants, distribution centers, or call centers) demand for the products or services can be assumed to be relatively independent of locations. However, location choices of stores such as coffee shops, supermarkets, bank branches, video clubs, and restaurants do have a direct impact on demand. Therefore, such decisions should not be made without demographics

and competition considerations. A multi-store retailer's location strategy formation usually follows a three-stage process: i) market selection, which involves the choice of regions or metropolitan areas for location, ii) trade area analysis, which evaluates the spatial structure of sub-regions in terms of market potential and level of competition, and iii) site evaluation, which focuses on the specific characteristics of individual sites (Ghosh and McLafferty 1987).

Various indices related to population demographics and income levels (hence, the market potential) are used in market selection decisions. The best known are the buying power index (BPI) and the premium-priced product (PPP) index of various metropolitan areas. The most commonly used index for the existing level of competition is known as the index of retail saturation (IRS), which is roughly the ratio of total retail expenditure to total square footage of retail establishments in an area. Several publications (such as Census of Retailing, Editor and Publisher Market Guide, Survey of Buying Power, and numerous others) provide these measures at various levels of geographic and industry detail. These indices are descriptive measures that can help retailers identify their most attractive markets.

Trade area analysis is conducted at a more detailed level and aims at identifying spatial variations within a market. Indices similar to those just described are used at a refined level to prioritize areas for expansion and to determine approximate level of expansion. Finally, site evaluation determines the most attractive locations among alternatives and involves detailed measures of local demographics, local retail structure, traffic flow and accessibility, site characteristics, legal issues, and cost factors (see Ghosh and McLafferty 1987, pp. 49 for a detailed checklist). Hence, precise locations of stores are determined after the firms determine their target markets and the degree of their entry or expansion. Furthermore, as we have just observed from the Second Cup and 7-Eleven Japan examples, firms actually apply various location strategies to contain entry or expansion by competitors, or to force them out of the market.

There exists quite an extensive and truly multi-disciplinary literature on competitive location. It ties in several fields including economics, marketing, operations research, and computational geometry. Each of these disciplines takes a different perspective of the problem. For example, the economics literature is mainly concerned with providing insights into competitive location strategies. These

models are relatively simple and apply predominantly to firms that own and operate single stores. In contrast, most studies published in the operations research literature are concerned with providing detailed solutions that can assist the managers' decision-making process. Most of these models are defined on networks where consumers and facilities are assumed to be on the nodes or along the edges. Operations research models provide detailed solutions but little insight. Moreover, they tend to require extensive data and computing times.

While these approaches contain important differences, they share certain common modeling assumptions. First, a majority of the models attempt to maximize the total revenue or market share under the assumption that the number of branches opened by the firms is exogenous. They also assume that firms are identical in terms of their cost structure. Although incorporating unit costs into the models is somewhat straightforward, fixed costs differences may pose challenges. Second, most of the models entail a very precise representation of the locations where new stores are to be opened. Although, such a construct may be valuable at the site evaluation level, we believe this is a fixation in pursuit of theoretical accuracy that has relatively little practical value in market selection or trade area analysis stages. Consider a case where a model suggests precise branch locations. There is quite a good chance that the firm will find some of these suggestions impossible to implement for many reasons that cannot be included in the model. Finally, it becomes much harder to solve these models as problem size increases. They become intractable either analytically or computationally.

In this article, we propose an alternative model for competitive location. It is based on an approximate rather than a precise representation of the store locations. In effect, we shift the issue from competition on location to competition on location density. The approximate location decisions are summarized in the definition of continuous functions. This methodology was introduced to the operations research literature in a seminal paper by Newell (1973) for a different class of problems, and referred to as "continuous approximation". It leads to intuitive models without sacrificing much of the substance. It has later been used in production-distribution system design (Geoffrion 1976, 1979), in general optimal market areas (Erlenkotter 1989), and in logistics system analysis (Daganzo 1991). To our knowledge, we are the first to apply it to competitive location.

We consider firms planning to open multiple stores in a geographical market. Demand at any branch is

not only a function of its location but also a function of other branch locations, whether owned by the same firm or by a competitor. Firms determine location strategies that maximize their profit under competition. Unlike the majority of past studies, we consider the number of stores to be opened endogenous by the introduction of fixed costs. Hence, our model resolves the tradeoffs involved in revenue maximization and fixed cost minimization while considering the location strategies of competitors. Two main parameters related to revenue and cost are the density of customers and fixed cost, which are represented as functions over the geographical space. At the outset, our model determines whether a firm should enter a market (or parts of it) or not. If the answer is affirmative, the model determines the approximate number of stores to be opened and their approximate locations. We leave the determination of the exact number and locations of these stores to a subsequent analysis, which would most likely include more detailed considerations as in site evaluation stage.

When contrasted with the three-stage decision making process described above, our model considers the strategic interactions in a situation where firms are at the market selection or trade area analysis stages. In our model, firms determine in which regions of the market they should enter and the approximate number and locations of stores they should open. We will show that these decisions can be represented by a single function, which we call location density function. This is a unique feature of our model that differentiates it from the past competitive location models. Furthermore, as opposed to earlier models, having a large number of potential stores in a market creates virtually no complication.

We develop our model starting with a precise representation of store locations. We then consider that firms know that they might not be able to locate exactly where they want. Furthermore, they do not heed much the exact locations of competing stores at those stages of their decision process. In addition, it may also happen that rival firms are also planning to open stores for which the sites are not yet finalized. We then apply the mechanics of continuous approximation to develop a model which is essentially a model of competition on location densities. Our consumer density and location density functions play roles comparable to that of indices such as the BPI and the IRS in market selection and trade area analysis. The difference, however, is that we use these functions in a formal game where firms maximize profits, whereas in reality firms use such indices mostly to compare and

prioritize regions in a market.

We should note that by shifting the strategy representation from exact locations to location density we most importantly eliminate the possibility of a phenomenon known as "sandwiching", which essentially refers to opening two new stores right around an existing rival store and capturing all of its demand. A typical location model would prescribe sandwiching if the entrant had a considerable cost advantage over the incumbent. As we will discuss later, some of our results rely on the absence of sandwiching in our model. There are numerous arguments against sandwiching, ranging from product and service heterogeneity to consumer search behavior. Our reasons are related to realities of location planning: as we have discussed, firms cannot always open at the exact location they had planned and furthermore the locations of rival firms may be unknown because they may also be in the process of location planning.

Before proceeding to our problem description we provide and overview of the literature on competitive location. For detailed reviews the reader may refer to Gabszewicz and Thisse (1986), Ghosh and McLafferty (1987), Eiselt, Laporte, and Thisse (1993), Kilkenny and Thisse (1999), Plastria (2001), and parts of Stole (2001).

The paper by Hotelling (1929) is widely recognized as the study that sets the stage in the economics literature. It considers the competition of two firms on a line segment of unit length. The line can represent a physical or a perceptual space in which the firms locate their facilities or products. Hotelling formulates a two-stage game, where firms choose their location in the first stage and compete on prices in the second stage. However, it is well known that under Hotelling's original assumptions, there may not be a pure-strategy price equilibrium (see for example, Gabszewicz and Thisse 1986). Several authors have later proposed refinements to the original Hotelling model. Some of these include zero conjectural variation axioms by Eaton (1972), random choice models by Anderson and de Palma (1992), a circular city by Salop (1979), quadratic transport costs by d'Aspremont, Gabszewicz, and Thisse (1979), convex transport costs by Economides (1986), vertical differentiation by Shaked and Sutton (1982), and spatial price discrimination by Lederer and Hurter (1986). Finally, Caplin and Nalebuff (1991) provide fairly general conditions for pure-strategy price equilibrium existence and uniqueness in an oligopoly.

Retail location has also been the target of much research in marketing. Estimating trade areas and market shares of retail branches is central to store location, relocation or dismantling, and to the management of retail networks. Simple models such as Voronoi tessellations and Reilly's law of retail gravitation have been around for much of the twentieth century (see for example, Ghosh and McLafferty 1987). These methods and their various extensions have been used to delineate trade areas for retail outlets, especially in the presence of limited data. For example Huff (1964, 1966) suggests a probabilistic choice model that considers distance and the relative attraction of locations. In addition, several computational geometry articles have proposed more comprehensive models using simple Voronoi tessellations. For example, Okabe and Suzuki (1987), Okabe and Aoyagi (1991), and Aoyagi and Okabe (1993) study pure-strategy location strategies of large number of single-branch firms in a two-dimensional space.

The operations research literature is generally concerned with competitive location on networks. Consumers are assumed to be located either at the nodes or spread uniformly along the edges of the network. Similarly, location choices of firms are restricted to nodes or edges. An important contribution in this area is due to Hakimi (1983) who formulates two problems under exogenously set prices. In the $(r|p)$-medianoid problem the follower locates r new facilities on a network where p competitor facilities are already located. In the $(r|p)$-centroid problem the leader locates p facilities so as to minimize the maximum profit of the subsequent r competitor facilities. Hakimi shows that these problems are NP-hard. Although, there has been some progress on the solution of medianoid problems (Megiddo, Zemel, and Hakimi 1983 and Dasci, Eiselt, and Laporte 2002) the solution of centroid problems remains quite challenging. To our knowledge, only the $(1|1)$-centroid problem has been solved (Hansen and Labbé 1988). Hakimi's work has also led to several extensions. These include maximum capture problems by ReVelle (1986) and Serra and ReVelle (1994), and the incorporation of different choice models by Serra et al. (1999). There are also several models on networks where firms locate simultaneously, such as oligopolistic competition with random consumer utility by de Palma et al. (1989), delivered pricing on networks by Lederer and Thisse (1990), and Cournot competition by Labbé and Hakimi (1991),

In all of the above models, the number of facilities firms can open is exogenously fixed. Models

in which the number of facilities is endogenously determined through fixed costs are provided by Dobson and Karmarkar (1987) and by Miller, Friesz, and Tobin (1996). Dobson and Karmarkar make various stable set definitions, i.e., the sets of store locations that the leader opens to deter entry by the follower. They present several algorithms to find these stable sets. Miller, Friesz, and Tobin describe a set of problems and present formulations to characterize Nash-Cournot and Stackelberg-Nash-Cournot equilibrium locations and production quantities, in which the leader determines the locations and the production quantities and the followers determine their production quantities but not the locations. They use variational inequality techniques to solve these problems.

The Dobson and Karmarkar model shares several features with ours. These authors have observed that a leader in the market can deter entry by preemptive store opening. However, in their model firms possess identical cost structures, the alternative store locations are already predetermined, and the follower is not allowed to co-locate. Therefore, when the leader's strategies are not constrained, the follower cannot open even a single profitable store because the worst the leader can do is to open a store at that node without lowering her profit. We, on the other hand, treat the consumers as entities continuously spread over the market according to a given density and consider any point in the space as an alternative store location for both firms. We also consider relevant costs to be represented as continuous functions defined over the market to account for geographical variations, as in Dobson and Karmarkar. However, we allow firms to have different fixed costs. On the revenue side, we assume that the marginal net revenue of firms (marginal revenue less marginal cost) is exogenous. Furthermore, we set these revenues equal without loss of generality.

In the interest of tractability, we impose several other restrictions, all of which have been extensively used in the past literature. We consider a bounded market in which firms open multiple branches. Each branch of a firm displays the same characteristics and therefore, there is horizontal competition between them, i.e., if branches of the same firm are too close, then they cannibalize each other's demand. We also assume that other store characteristics such as service quality, merchandise assortment, advertising, price, and promotions are exogenously fixed and do not affect the competition. Therefore, consumers are assumed to patronize the closest store. The model could realistically represent many industries given the high degree of franchising and the resulting standardization among sister

branches. However, an extension of this model should also consider the firms' strategic choices for some of these characteristics.

We consider both one- and two-dimensional markets but the presentation of the model and the derivation of the main results are conducted in the one-dimensional case for the sake of simplicity. We should note, however, that in certain cases (such as competition on a main road or along a highway) a one-dimensional market can be an accurate representation of reality. Nevertheless, we extend parts of the analysis to the two-dimensional case and show that all of the results remain valid under mild conditions.

Our analysis provides several insights into the location strategies of firms under competition. First, we establish that if the follower decides to enter a market, he enters at least as strongly as the leader. Second, a market leader can effectively deter entry even if she is severely cost disadvantaged. We have found that even if the leader's fixed costs are twice those of the follower, she can open enough stores to deter entry and still make a profit. However, in some of those cases the leader is better off by allowing the follower to enter the market. Third, the leader may also let a follower enter and take at least half of the revenues in some cases where she is extremely cost advantaged. We have also shown that once both firms enter the market, their location strategies become rather insensitive to small changes in cost and demand parameters.

The remainder of this article is organized as follows. In the next section we introduce additional assumptions, variable definitions, and the methodology on a one-dimensional market. Section 3 analyzes the location strategy of the follower and Section 4 analyzes that of the leader. A comparison of results of a competition on location and competition on location density is provided in Section 5. Section 6 extends the model to two-dimensional markets. Concluding remarks and future research directions are provided in Section 7.

Figure 1: Consumer utility in a five-store example.

2 Continuous Approximation on a Linear Market

We now present our model on a linear market where there is only one firm facing no competition, thus laying the groundwork for the subsequent sections. Some of the basic notation and the assumptions are also presented here. Let the consumers be dispersed on the line segment $[0, 1]$ according to a probability density function $f(x)$. In reality, this function has to be constructed from discrete data available at the zip code or metropolitan statistical area level. The interested reader may refer to Donthu and Rust (1989) for a kernel density estimation procedure and a review of other methods on how to smooth sample data into a continuous probability density function. Each customer who decides to purchase, purchases one unit of a product or service. The total attainable revenue is normalized to unity without loss of generality, i.e.,

$$\int_0^1 f(x)dx = 1.$$

The firm has n branches located at $x_1, x_2, \ldots, x_n \in [0, 1]$. Consumer $x \in [0, 1]$ derives a utility $u_i(x)$ by patronizing branch i, given as:

$$u_i(x) = a - b|x - x_i|.$$

We assume that a and b are the same for all consumers. A consumer patronizes the facility that provides the largest positive utility. Hence consumer x patronizes store i if

$$u_i(x) \geq u_j(x), \quad \forall j, \text{ and } u_i(x) \geq 0$$

Let $u(x)$ denote the maximum utility (i.e., intrinsic utility) consumer x derives. Then,

$$u(x) = \max_i u_i(x).$$

Figure 1 depicts the maximum attainable utility for each consumer in an example where the firm has five stores. Note that all consumers, except a group between the second and the third stores, derive positive utility. Therefore, they patronize the closest store, i.e., they are captured by the firm. Let $1_{\{u(x)\geq 0\}}(x)$ be an indicator function denoting whether consumer x is captured or not. Hence, the total revenue can be written as:

$$\int_0^1 1_{\{u(x)\geq 0\}}(x) f(x) dx, \tag{1}$$

where, for each $x \in (x_i, x_{i+1}]$,

$$1_{\{u(x)\geq 0\}}(x) = \begin{cases} 1 & \text{if } x \leq x_i + a/b \\ 1 & \text{if } x \geq x_{i+1} - a/b \\ 0 & \text{if } x_i + a/b < x < x_{i+1} - a/b. \end{cases}$$

To deal with both ends of the line segment we define pseudo-stores at $x_0 = -x_1$ and $x_{n+1} = 2 - x_n$. We now turn our attention to costs in order to determine the total profit. Without loss of generality, we assume that the unit cost of servicing has already been deducted from the unit revenue. Therefore, (1) denotes the net total revenue and the only cost element to be considered is the total fixed cost. Let $K(x)$ denote the fixed cost of opening a branch at location x. Given branch locations $x_1, \ldots, x_n$, the total fixed cost is given as $\sum_j K(x_j)$. Therefore, the problem can be formulated as:

$$\max_{0\leq x_1\leq \ldots \leq x_n \leq 1} \int_0^1 1_{\{u(x)\geq 0\}}(x) f(x) dx - \sum_j K(x_j). \tag{2}$$

One possible way to solve the above problem is to fix n and then devise a search procedure over the x_i's. However, the problem is nonlinear and nonconvex in general. Even though efficient solution algorithms could conceivably be obtained for special cases, we doubt whether the effort is worth the outcome. Instead, we follow an alternative approach based on the approximation of the net total revenue and total cost.

Approximating the net total revenue basically follows a probabilistic analysis of the intrinsic utility, which is essentially deterministic. Consider an *arbitrary* consumer $x \in (x_i, x_{i+1}]$ for which $z(x)$ is a random variable denoting the minimum distance from x to one of the branches. Clearly, $u(x) = a - bz(x)$ is also a random variable denoting the intrinsic utility. Let $Q(x)$ denote the probability that

consumer x patronizes one of the stores, i.e., $Q(x) = P\{u(x) \geq 0\}$. Therefore, the expected total revenue is:

$$\int_0^1 Q(x)f(x)dx. \tag{3}$$

We now determine the *approximate* distribution of $z(x)$ and hence, that of $u(x)$. Note that, for any $x \in (x_i, x_{i+1}]$,

$$P\{z(x) \geq z\} = 0 \quad \text{if } z \geq (x_{i+1} - x_i)/2.$$

For $z \leq (x_{i+1} - x_i)/2$,

$$P\{z(x) \geq z\} = \frac{\int_{x_i+z}^{x_{i+1}-z} f(x)dx}{\int_{x_i}^{x_{i+1}} f(x)dx}.$$

If consumer density is linear in the interval $(x_i, x_{i+1}]$ (i.e., locally linear) then,

$$P\{z(x) \geq z\} = 1 - \frac{2z}{x_{i+1} - x_i},$$

i.e., $z(x)$ is uniform. Since consumer density is not linear in general, $z(x)$ is only approximately uniform. Similarly, $u(x)$ also follows an approximately uniform distribution between $[a - b(x_{i+1} - x_i)/2, a]$. Finally,

$$Q(x) = \begin{cases} 1 & \text{if } x_{i+1} - x_i \leq 2a/b \\ 2a/b(x_{i+1} - x_i) & \text{otherwise.} \end{cases} \tag{4}$$

Note that if the distance between two neighbor branches is less than a certain threshold, all consumers between them are captured by the firm. However, if the neighbor stores are sufficiently far apart, some consumers in the middle will be lost. The assumption of local linearity helps us express this loss as a function of the distance between the stores and average it out across the interval. In order to simplify the expressions we introduce a step function $g_s(x)$ as

$$g_s(x) = x_{i+1} - x_i \text{ for } x \in (x_i, x_{i+1}].$$

Note that no information is lost by the introduction of the step function because there is a one-to-one relationship between the store locations and the step function. An example of a step function is depicted in Figure 2. Hence, for all $x \in [0, 1]$,

$$\begin{aligned} z(x) &\sim \text{Uniform}[0, g_s(x)/2], \\ u(x) &\sim \text{Uniform } [a - bg_s(x)/2, a], \quad \text{and} \end{aligned}$$

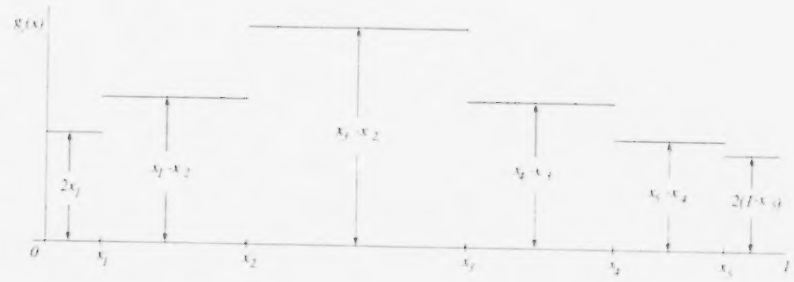

Figure 2: An example of a step function

$$Q(x) = \begin{cases} 1 & \text{if } g_s(x) \leq 2a/b \\ 2a/bg_s(x) & \text{otherwise.} \end{cases}$$

As will be seen, the role of this step function goes beyond the simplification of the formulation. We now express the total fixed cost in terms of the step function $g_s(x)$ defined above. First, consider the following identity:

$$\sum_i K(x_i) = \sum_i K(x_i) \left[\int_{(x_i+x_{i-1})/2}^{x_i} \frac{dx}{x_i - x_{i-1}} + \int_{x_i}^{(x_{i+1}+x_i)/2} \frac{dx}{x_{i+1} - x_i} \right].$$

Note that each of the integrals in the brackets is equal to 1/2, i.e., we multiply each $K(x_i)$ by one. Using the definition of the step function $g_s(x)$, this expression can be rewritten as:

$$\begin{aligned} \sum_i K(x_i) &= \sum_i K(x_i) \left[\int_{(x_i+x_{i-1})/2}^{x_i} \frac{dx}{g_s(x)} + \int_{x_i}^{(x_{i+1}+x_i)/2} \frac{dx}{g_s(x)} \right] \\ &= \sum_i K(x_i) \int_{(x_i+x_{i-1})/2}^{(x_{i+1}+x_i)/2} \frac{dx}{g_s(x)}. \end{aligned}$$

Similarly, we assume that fixed costs are locally linear in these intervals. In effect, $K(x)$ can be perceived as a slowly varying function in x. This assumption allows us to express the total fixed cost as a single integral. Thus,

$$\begin{aligned} \sum_i K(x_i) &\approx \sum_i \int_{(x_i+x_{i-1})/2}^{(x_{i+1}+x_i)/2} \frac{K(x)dx}{g_s(x)} \quad \text{(because } K(x) \text{ is slowly varying)} \\ &= \int_0^1 \frac{K(x)dx}{g_s(x)}. \end{aligned}$$

Given the number of branches, the problem of maximizing the total profit can be rewritten as:

$$\max_{g_s(x) \geq 0} \Pi[g_s(x)] = \int_0^1 \left[Q(x)f(x) - \frac{K(x)}{g_s(x)} \right] dx. \tag{5}$$

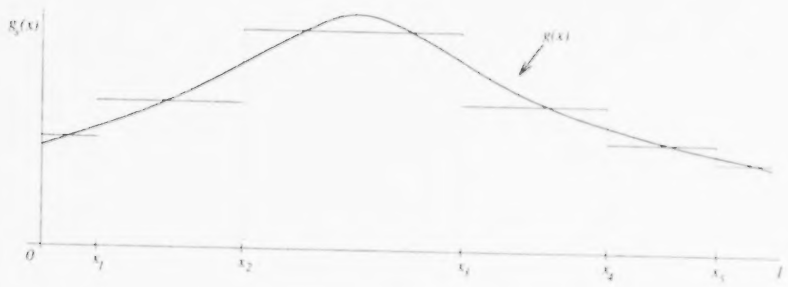

Figure 3: A step function and its continuous approximate

To formulate the problem, we have assumed that both the consumer density and fixed cost change slowly in close neighborhoods, i.e., they are locally linear in the intervals. This assumption is critical for the analysis and is often justified. We observe that important factors of consumer density and fixed costs, such as demographics and real estate prices, do not display abrupt changes in close neighborhoods. Even if they do, it is better to consider them in a more detailed subsequent analysis rather than in a high level analysis such as this.

In any case, solving (5) is akin to solving (2). Therefore, rather than trying to find a step function and hence, the precise branch locations, we first find a smooth function $g(x)$ that solves (5), and then a step function $g_s(x)$ consistent with its smooth counterpart can be found by a subsequent analysis. This is the idea behind the continuous approximation approach (see Newell 1973). An example of $g_s(x)$ and $g(x)$ is depicted in Figure 3. Hence, the alternative formulation is:

$$\max_{g(x)\geq 0} \Pi[g(x)] = \int_0^1 \left[Q(x)f(x) - \frac{K(x)}{g(x)} \right] dx, \tag{6}$$

where

$$Q(x) = \begin{cases} 1 & \text{if } g(x) \leq 2a/b \\ 2a/bg(x) & \text{otherwise.} \end{cases}$$

Similarly,

$$\begin{aligned} z(x) &\sim \text{Uniform}[0, g(x)/2], \\ u(x) &\sim \text{Uniform}\,[a - bg(x)/2, a]. \end{aligned}$$

Several remarks are in order here. First, the number of branches is already incorporated in the new formulation. Therefore, we do not need to perform a search over n to solve the optimization

problem. Second, the smooth function approximates the distance between neighbor branches. From this function, we can obtain the approximate store locations and their trade areas. Third, it is clear that such an approximation is quite accurate provided the number of branches is large. Note that the accuracy levels of the two approximations are intimately related. An increase in the possible number of branches leads to higher accuracy not only of the second approximation, but also of the first one because the intervals on which $f(x)$ and $K(x)$ need to be locally linear become smaller. Therefore, the methodology becomes more accurate as the problem size (i.e., the number of branches) increases, leaving the problem complexity virtually unchanged. However, the continuous approximation remains a heuristic because it ignores integrality. Therefore, even if consumer density or fixed cost were uniform across the market, there would still be an error in the analysis.

Solving (6) is a simple exercise given the problem parameters. Basically, the firm divides the market into two regions, profitable and non-profitable, and opens equidistant stores in the profitable regions. Hence, beyond a threshold value that depends on the firm's fixed cost, consumer density has no impact on the dispersion of the stores. One obvious reason for this is the inelastic demand assumption we have made, i.e., any consumer with a non-negative utility is captured by the firm. Another obvious reason is the fixed retail price assumption of the model. A less obvious reason, however, is the absence of rival stores. In the forthcoming sections, we show that consumer density $f(x)$ has a secondary impact on the firms' location strategies under competition, even under the fixed price assumption. A similar reasoning applies to the fixed cost, i.e., below a certain threshold value that depends on consumer density, it has no impact on the firm's location strategy, but it has an impact under competition.

We close this section with a comment on the relationship between the step and smooth functions. Given a smooth function, a step function can be found by solving

$$\min_{g_s(x)} S = \int_0^1 |g(x) - g_s(x)| dx,$$

where $g_s(x)$ belongs to a special set of step functions described above (e.g., as depicted in Figure 2). It is easy to show that the above minimization problem is well defined and therefore, admits a unique optimal step function. Solving this minimization problem, however, is extremely difficult as it contains many local extrema. Furthermore, the optimal step function is not necessarily optimal for the original problem. However, a step function can give an idea about where the stores should

Step 0:	Form $n(x)$ from $g(x)$.
Step 1:	Set $n = 1$.
Step 2:	For $i = 1, \ldots, n$, assign the $(2i-1)/(2n)$th fractile of $n(x)$ to x_i.
Step 3:	Construct the step function and compute the objective function S.
Step 4:	If S improves, increment n by 1 and go to Step 2.
	Else, decrement n by 1 and go to Step 5.
Step 5:	For $i = 1, \ldots, n$, assign the $(2i-1)/(2n)$th fractile of $n(x)$ to x_i.
Step 6:	Stop.

Figure 4: Description of the heuristic to derive a step function from a smooth function

be located and hence assist a detailed site analysis stage. Daganzo (pp. 75-78, 1991) conjectures and informally shows that a heuristic treatment is acceptable and he presents such a heuristic. The heuristic we have envisioned is somewhat different and produces better results than that of Daganzo. It is based on the following observation: after we obtain $g(x)$, defined on $[0, 1]$, we define a location density function as $\bar{n}(x) = 1/g(x)$ and we normalize it to a probability density function $n(x)$; we then essentially discretize this function. The heuristic is given in Figure 4. This procedure is trivial to code and also quite accurate. We have tested it on the smooth functions that we have obtained in the numerical experiments summarized at the end of the next section. Only to give the reader an idea, the average value of $g(x)$ in all of our experiments is 0.057 and the average value of S is 0.001. These figures suggest that our procedure is quite satisfactory for all practical purposes.

3 The Follower's Problem

In this section we analyze the location strategies of a firm, the follower, planning to open a number of stores in a market already served by another firm, the leader. As in the previous section, we start with precise representations of the firms' store locations and then apply the continuous approximation to reformulate the problem. As a result the problem is transformed from one of competition on location to one of competition on location density.

The previous notation remains valid but subscripts and superscripts 0 and 1 are used to denote

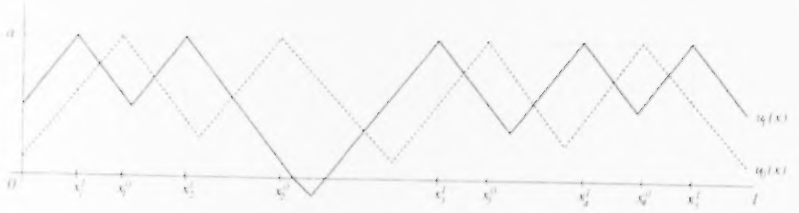

Figure 5: An example of consumer utilities when there are competing facilities.

the leader and the follower, respectively. An example solution is depicted in Figure 5, in which the follower has five stores denoted by x_i^1, $i = 1, 2, 3, 4$, and 5, and the leader has four stores denoted by x_j^0, $j = 1, 2, 3$, and 4. Functions $u_1(x)$ and $u_0(x)$ denote the intrinsic utilities a consumer x derives from the follower and the leader respectively. Similarly, $z_1(x)$ and $z_0(x)$ denote the minimum distance from the consumer x to follower's and leader's stores. We assume that both the leader and follower provide a very standardized product or service, i.e., a and b are the same for all firms (not only for branches that belong to a firm). This assumption can easily be relaxed albeit at the expense of analytical simplicity. Treating each firm separately, we can show that for an arbitrary consumer $x \in (x_i^1, x_{i+1}^1]$,

$$P(z_1(x) \geq z) = \begin{cases} 0 & \text{if } z \geq (x_{i+1}^1 - x_i^1)/2 \\ \dfrac{\int_{x_i^1+z}^{x_{i+1}^1 - z} f(x)dx}{\int_{x_i^1}^{x_{i+1}^1} f(x)dx} & \text{otherwise,} \end{cases} \tag{7}$$

and for an arbitrary consumer $x \in (x_j^0, x_{j+1}^0]$,

$$P(z_0(x) \geq z) = \begin{cases} 0 & \text{if } z \geq (x_{j+1}^1 - x_j^1)/2 \\ \dfrac{\int_{x_j^0+z}^{x_{j+1}^0 - z} f(x)dx}{\int_{x_j^0}^{x_{j+1}^0} f(x)dx} & \text{otherwise.} \end{cases} \tag{8}$$

As an immediate extension of the previous section, let $Q_1(x)$ denote the probability that the follower captures consumer x. Then,

$$Q_1(x) = P\{u_1(x) \geq u_0(x)\} \tag{9}$$

$$= P\{z_1(x) \leq z_0(x)\}. \tag{10}$$

Hence, given the number of stores n, the follower solves:

$$\max_{0\le x_1^1\le\cdots\le x_n^1\le 1}\int_0^1 Q_1(x)f(x)dx-\sum_j K(x_j^1). \tag{11}$$

This problem is a very difficult nonlinear program with possibly a large number of local maxima. Therefore, we repeat the treatment presented in the previous section to reach a more tractable model, i.e., we first approximate the probability distributions with uniform distributions and remodel the problem with the step functions and finally approximate the step functions with their smooth counterparts. Hence, as we have established in the previous section, given the leader's and the follower's location strategies, i.e., smooth functions $g_0(x)$ and $g_1(x)$, the minimum distance and the maximum utility an arbitrary consumer x receives from the firms can be represented as random variables with the following distributions:

$$z_0(x)\sim \text{Uniform}[0,g_0(x)/2] \text{ and } u_0(x)\sim \text{Uniform } [a-bg_0(x)/2,a]$$

and

$$z_1(x)\sim \text{Uniform } [0,g_1(x)/2] \text{ and } u_1(x)\sim \text{Uniform } [a-bg_1(x)/2,a].$$

The follower first decides whether to enter the market or not. He decides on $g_1(x)$ if the entry decision is positive. These decisions can easily be combined in the definition of the smooth function where $g_1(x)=\infty$ represents the no-entry decision in the part of the market that x belongs to. Let $Q_1(x)$ denote the probability that the follower captures consumer x. Then,

$$\begin{aligned}
Q_1(x) &= P\{z_1(x)\le z_0(x)\}\\
&= \int_0^{g_0(x)/2} P\{z_1(x)\le z_0(x)|z_0(x)=z\}dF_{z_0(x)}(z)\\
&= \int_0^{g_0(x)/2} P\{z_1(x)\le z\}\frac{2}{g_0(x)}dz\\
&= \begin{cases} \frac{g_0(x)}{2g_1(x)} & \text{if } g_1(x)\ge g_0(x)\\ 1-\frac{g_1(x)}{2g_0(x)} & \text{if } g_1(x)\le g_0(x). \end{cases}
\end{aligned} \tag{12}$$

The above treatment is valid if the follower is assumed to know the location strategy of the leader (which is represented as a continuous function), but not the precise locations of its outlets. This is precisely the case when firms are making market selection decisions and trade area analyses. Furthermore, we have no presumption of any of the leader's stores to be set up before those of the follower.

However, the leader is assumed to correctly and convincingly signal her location strategy to the follower. She could accomplish this by revealing her location strategies in other markets or by opening few of the stores in parts of this market.

To illustrate, suppose that for all $x \in [0,1]$, $g_0 = 0.1$ and $g_1 = 0.2$, i.e., the leader is expected to have ten equidistant stores whereas the follower has five. Then equation (12) suggests that the follower captures 25% of the market. If $g_0 = 0.2$ and $g_1 = 0.1$, then the follower captures 75%, if $g_0 = 0.1$ and $g_1 = 0.01$ then the follower captures 95%, and so on. Hence, our formulation does not make any provision for sandwiching. That is, in the last two cases, if the follower had known the exact locations of the leader's stores and was able to locate exactly where he wants, he would have captured the entire market by sandwiching the leader's stores. Therefore, competition on location density is somewhat different from a standard location competition such as the one presented in Dobson and Karmarkar (1987). Hence, the follower solves the following problem:

$$\max_{g_1(x)\geq 0} \Pi_1[g_1(x), g_0(x)] = \int_0^1 \left[Q_1(x)f(x) - \frac{K_1(x)}{g_1(x)}\right] dx. \tag{13}$$

A function that maximizes the integral maximizes the integrand at every point when the integrand does not contain any integral or derivative of the decision variables. Therefore, we need to solve the following for each x:

$$\max_{g_1(x)\geq 0} \pi_1[g_1(x), g_0(x)] = Q_1(x)f(x) - \frac{K_1(x)}{g_1(x)}.$$

There are two cases:

Case 1: Given $g_1(x) \geq g_0(x)$, the follower solves

$$\max_{g_1(x)\geq g_0(x)} \pi_1[g_1(x), g_0(x)] = \frac{g_0(x)f(x)}{2g_1(x)} - \frac{K_1(x)}{g_1(x)},$$

Case 2: Given $g_1(x) \leq g_0(x)$, the follower solves

$$\max_{g_1(x)\leq g_0(x)} \pi_1[g_1(x), g_0(x)] = f(x) - \left(\frac{g_1(x)f(x)}{2g_0(x)} + \frac{K_1(x)}{g_1(x)}\right).$$

Let

$$\bar{g}(x) = \left(\frac{2g_0(x)K_1(x)}{f(x)}\right)^{1/2}.$$

If $\bar{g}(x) \leq g_0(x)$ then $g_0(x)f(x)/2 - K_1(x) \geq 0$, i.e., the same condition in as Case 1. Hence, we can summarize the entrant's best strategy as:

$$g_1^*(x) = \begin{cases} \bar{g}(x) & \text{if } g_0(x)f(x)/2 - K_1(x) \geq 0 \\ \infty & \text{otherwise,} \end{cases} \tag{14}$$

for an expected profit of:

$$\Pi_1[g_1^*(x), g_0(x)] = \int_0^1 \pi_1^*(x)dx$$

where

$$\pi_1^*(x) = \begin{cases} f(x) - (2K_1(x)f(x)/g_0(x))^{1/2} & \text{if } g_0(x)f(x)/2 - K_1(x) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

This is the main result of this section. It essentially suggests that the firm will not enter parts of the market where it cannot recover its fixed costs. If the follower decides to enter a market (or a subset of the market) he will do so by opening at least as many stores as the existing competition has and hence, will capture at least half of the market. This statement may be meaningless in certain settings because the follower can be declared to open more stores than the leader in any small interval around his stores. But once one considers the system under more restrictive assumptions of uniform consumer density and constant fixed costs across the market, the statement simply suggests that the follower opens more stores and gets more than half of the revenue. A second result from the above analysis is that once the entry conditions are satisfied, the follower's location density strategy is somewhat robust to small changes in consumer density and fixed cost.

Both of these results rely critically on the lack of sandwiching on the follower's part. Although the follower's entry conditions are somewhat different in location density competition than ir cation competition (as we will show in Section 5), if entry occurs in location competition, th ollower captures the entire market where it enters and the number of stores he opens become quite sensitive to the *leader's* fixed cost.

In the model, we have assumed that firms offer a very standardized service. Therefore, we hold a and b constant for all branches regardless of their owners. It is straightforward to extend the analysis to different a and b's but the solution might not come out in closed form. We have also assumed that the market is already fully served by the leader, i.e., $a - bg_0(x)/2 \geq 0$ for all $x \in [0, 1]$, i.e., the follower

Figure 6: Density function (a) and the solution (b) for the example.

is fighting for a share of an already captured market. But it could also be relaxed by dividing the market into two parts: those that are captured and those that are not. Then, we would employ the analysis presented in the previous section for the parts that are not captured. We have also neglected possible economies of scale resulting from having a large presence in a region. Usually, firms benefit from economies of scale in the areas of regional management, distribution, advertisement, and work force training when they have a larger presence in a market. Our result suggests that even without those extensive scale economies, there is an incentive for the firm to enter the market as strongly as the leader.

We now solve a simple illustrative example to demonstrate how our methodology works. To this end, consider a simple triangular consumer density given as:

$$f(x) = \begin{cases} 4x & \text{if } 0 \le x \le 1/2 \\ 4(1-x) & \text{if } 1/2 \le x \le 1, \end{cases}$$

and depicted in Figure 6(a). Assume that there are ten existing stores perhaps equally spaced over the market, that is $g_0(x) = 0.1$ for all $x \in [0,1]$. Also, let $K_1(x) = 0.05$ for all $x \in [0,1]$. Then the optimal location strategy of the entrant can be computed as:

$$g_1^*(x) = \begin{cases} \infty & \text{if } 0 \le x < 1/4 \\ 0.05/\sqrt{x} & \text{if } 1/4 \le x \le 1/2 \\ 0.05/\sqrt{1-x} & \text{if } 1/2 \le x \le 3/4 \\ \infty & \text{if } 3/4 < x \le 1. \end{cases}$$

Figure 6(b) depicts the solution of this example. A simple computation reveals that the follower will open around six branches (i.e., $\int_0^1 dx/g_1(x) = 6.10$) in the middle segment as compared to five existing

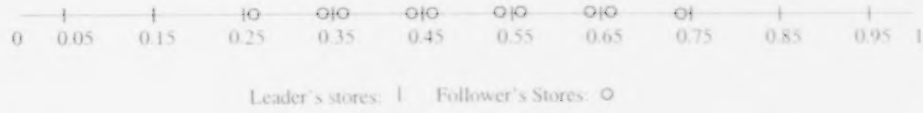

Figure 7. Follower's solution on location competition.

branches. On the other hand, he will not open any stores in the first and fourth quartiles, because he cannot recover his fixed costs in those regions, i.e.,

$$g_0(x)f(x)/2 - K_1(x) = \begin{cases} 0.2x - 0.05 \leq 0 & \text{if } 0 \leq x < 1/4 \\ 0.2(1-x) - 0.05 \leq 0 & \text{if } 3/4 \leq x \leq 1. \end{cases}$$

The expected profit of this strategy is:

$$\int_{1/4}^{3/4} \left[f(x) - (2K_1 f(x)/g_0(x))^{1/2} \right] dx = 0.0703.$$

To highlight the difference between competition on location and competition on location density, assume that the leader has ten equidistant stores (at $0.05, 0.15, \ldots, 0.95$) and the follower has complete knowledge of these locations. Then the follower's strategy will be remarkably different. As shown in Figure 7, the follower opens ten stores in [0.25,0.75] sandwiching some of leader's stores and capturing all the demand in that interval for a total profit of 0.167.

In the remainder of this section, we describe the problem set that we have used to assess the accuracy of the heuristic presented in Figure 4. The problem set is constructed as follows:

- the geographic market is normalized to the line segment [0,1];
- the consumer density is assumed to have a truncated exponential distribution, i.e., $f(x) = (\lambda_f e^{-\lambda_f x})/(1 - e^{-\lambda_f})$. Hence, the total revenue in the market is normalized to one;
- the fixed cost is also set to equal a negative exponential function $K(x) = \kappa e^{-\lambda_K x}$;
- the density of the existing stores is set equal to a linear function $g_0(x) = a + bx$.

The highest and lowest values of consumer density and fixed cost are controlled through λ_f and λ_K. Parameter κ is used to control the magnitude of the fixed cost. We have considered five values for

λ_f and λ_K given approximately as 0.69, 1.10, 1.61, 2.30, and 2.99. These values imply that the ratio of the maximum value to the minimum value of the parameters is 2, 3, 5, 10, and 20, respectively. The primary reason for our selection of exponential functions is that it is relatively easy to control the degrees of local linearity and the geographical variation in the problem by these functions. In the density function of the leader's stores, we set $b = 0.1$ and change the values of a so that the number of existing stores can be 2, 3, or 5. The value of κ is set to 0.01 to ensure that the follower opens stores everywhere in the market, i.e., $g_1(x) < \infty$ for all $x \in [0, 1]$. This restriction is imposed only to simplify the coding not due to a conceptual difficulty. Hence, the total number of instances is 75, for which the follower's best response can be found from (14) as:

$$g_1^*(x) = \left(\frac{2(a + bx)(\kappa e^{-\lambda_K x})}{(\lambda_f e^{-\lambda_f x})/(1 - e^{-\lambda_f})} \right)^{1/2}.$$

We then obtained step functions for these instances using the heuristic. We consider the ratio of S to the average of $g_1^*(x)$ as a measure of the heuristic's accuracy. In 75 instances, we have found that the ratio is about 1.5% on the average with a maximum of 3.9%.

4 Duopoly: Stackelberg-Nash Equilibrium

We now extend the analysis to competition between two firms, where one of them has an advantage over the other. The advantage comes from being the first mover as well as having prior knowledge of the subsequent arrival of a new competitor. This leader-follower game is assumed to be one of perfect information on location density decisions and of complete information on all consumer density and cost parameters. For the sake of clarity, we assume that there are no existing stores that belong to other firms. Relaxing this assumption, however, is rather easy. The sequence of the game is as follows:

Stage 1: The leader chooses $g_0(x)$.

Stage 2: Having observed $g_0(x)$, the follower chooses $g_1(x)$.

In Stage 2, we do not explicitly model the entry decision because, as described in the previous section, $g_1(x) = \infty$ indicates a no-entry decision. The usual approach in this game is to start from Stage 2 and

find the response function of the follower for any strategy the leader may choose, which is analyzed in the previous section. The follower's best strategy is given by:

$$g_1^*(x) = \begin{cases} \left(\frac{2g_0(x)K_1(x)}{f(x)}\right)^{1/2} & \text{if } g_0(x)f(x)/2 - K_1(x) \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

Let us first assume that the leader wants to deter entry. The most economical way for her to do so is to choose:

$$g_0^*(x) = \frac{2K_1(x)}{f(x)} - \epsilon, \tag{15}$$

where, ϵ is an arbitrarily small positive number. However, in order to simplify the notation, we avoid ϵ-strategies and assume that ϵ is zero. Hence, the follower sets $g_1^*(x) = \infty$ for all $x \in [0,1]$ and the leader captures the entire market, i.e., $Q_0(x) = 1$. Then, her total profit is given by:

$$\begin{aligned} \Pi_0[g_0^*(x), \infty] &= \int_0^1 \left[Q_0(x)f(x) - \frac{K_0(x)}{g_0^*(x)}\right] dx \\ &= \int_0^1 \left[f(x) - f(x)\frac{K_0(x)}{2K_1(x)}\right] dx \\ &= \int_0^1 \left[1 - \frac{K_0(x)}{2K_1(x)}\right] f(x)dx. \end{aligned}$$

Hence, a sufficient condition for the leader to deter entry is to have

$$K_0(x) \leq 2K_1(x), \text{ for all } x \in [0,1], \tag{16}$$

so that her profit becomes positive. It should be noted that this condition is not necessary. There could be less restrictive cases where the leader can deter entry, depending on the consumer density and the values of fixed costs. Then, clearly, the leader incurs a loss in certain regions but covers those losses with the profits from other areas. This shows that a leader can retain her monopoly position even though she is considerably cost disadvantaged. As we have just shown, the leader's fixed costs could be more than twice those of the follower, yet she could stay as a monopoly.

A natural question is whether the leader should pursue such a strategy at all costs. Let us now consider the case where the leader allows entry with the foreknowledge that she will lose at least half of the market to the follower. The leader sets $g_0(x)$ such that the follower enters the market (this is the case where $g_0(x)f(x)/2 - K_1(x) \geq 0$). Then,

$$g_1^*(x) = \left(\frac{2g_0(x)K_1(x)}{f(x)}\right)^{1/2}, \tag{17}$$

and

$$Q_0(x) = \left(\frac{K_1(x)}{2f(x)g_0(x)}\right)^{1/2}.$$

Hence, the leader's problem is

$$\begin{aligned}\max_{g_0(x)\geq 2K_1(x)/f(x)} \Pi_0[g_0(x), g_1^*(x)] &= \int_0^1 \left[Q_0(x)f(x) - \frac{K_0(x)}{g_0(x)}\right] dx, \\ &= \int_0^1 \left[f(x)\left(\frac{K_1(x)}{2f(x)g_0(x)}\right)^{1/2} - \frac{K_0(x)}{g_0(x)}\right] dx, \\ &= \int_0^1 \left[\left(\frac{K_1(x)f(x)}{2g_0(x)}\right)^{1/2} - \frac{K_0(x)}{g_0(x)}\right] dx. \end{aligned} \tag{18}$$

In the Appendix we show that the integrand is quasi-concave in $g_0(x)$ for all x. Therefore, the first order condition would provide the optimal solution given as:

$$g_0^*(x) = \left(\frac{8}{K_1(x)f(x)}\right)^{1/4}\sqrt{K_0(x)}, \tag{19}$$

provided $g_0^*(x) \geq 2K_1(x)/f(x)$. Hence, the follower may enter only if

$$\left(\frac{8}{K_1(x)f(x)}\right)^{1/4}\sqrt{K_0(x)} \geq 2K_1(x)/f(x),$$

i.e.,

$$K_0(x) \geq \sqrt{\frac{K_1(x)^5}{2f(x)^3}}. \tag{20}$$

The leader's profit is then:

$$\Pi_0[g_0^*(x), g_1^*(x)] = \int_0^1 \left(\frac{K_1(x)f(x)}{K_0(x)}\right)^{1/4}\left[\left(\frac{K_1(x)f(x)}{2}\right)^{3/8}\frac{1}{\sqrt{2}} - \left(\frac{K_0(x)}{2}\right)^{3/4}\right] dx.$$

A sufficient condition for the leader to have positive profit under entry is expressed as:

$$\sqrt[3]{2}K_0(x)^2 \leq K_1(x)f(x), \text{ for all } x \in [0,1], \tag{21}$$

i.e., the second term in the above integrand is positive. Now suppose $K_0(x) \leq 2K_1(x)$. Then the leader prefers to monopolize if

$$\left[1 - \frac{K_0(x)}{2K_1(x)}\right] f(x) \geq \left(\frac{K_1(x)f(x)}{K_0(x)}\right)^{1/4}\left[\left(\frac{K_1(x)f(x)}{2}\right)^{3/8}\frac{1}{\sqrt{2}} - \left(\frac{K_0(x)}{2}\right)^{3/4}\right]. \tag{22}$$

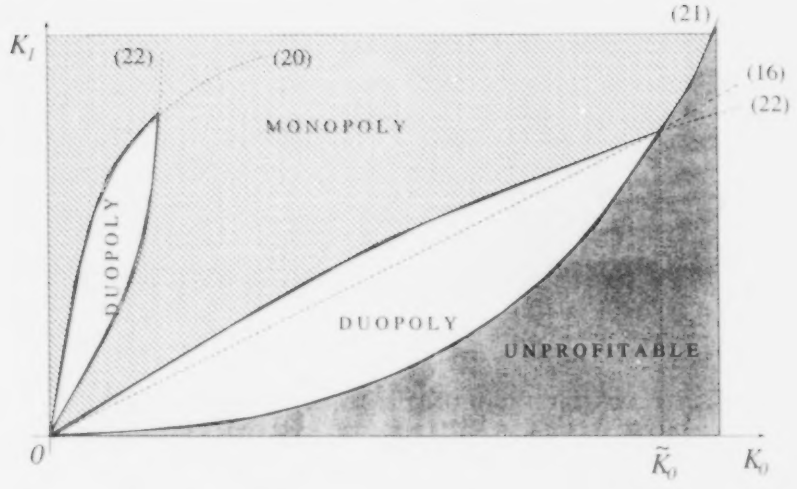

Figure 8: Range of fixed costs where the equilibrium output changes.

Therefore, depending on the values of $\tilde{K}_0(x)$, $K_1(x)$, and $f(x)$, there are regions where the leader, the follower, or both can be present in the market. Conditions (16), (20), (21), and (22) are depicted in Figure 8, where

$$\tilde{K}_0(x) = f(x)/2\sqrt[3]{2}. \tag{23}$$

Dashed lines denote the boundaries of the conditions while solid lines denote the active ones. Note that in some regions of parameters the leader prefers duopoly to monopoly even though she could make positive profits by opening more stores and hence preventing entry.

It is clear that the leader has an undisputed first mover advantage. For example, she could monopolize the market even if she is considerably cost disadvantaged. Moreover, she could still retain some market and hence, make a profit even when the advantage of the follower is severe. There is a certain region of fixed cost combinations where the leader does not enter the market. Hence, in order to have a game, condition (21) must be satisfied. Observe from the figure that in some cases, the leader has considerable advantage over the follower yet she prefers an entry by the follower rather than monopolization. However, we are unable to prove this for all values of $f(x)$ due to the intractability of condition (22). Furthermore, these cases are characterized by very small fixed costs (smaller than the figure would indicate).

UNPROFITABLE

A referee has pointed out that the last result is an example of a "judo strategy". Inspired by the pioneering work of two economists (Gelman and Salop 1983) who coined the term "judo economics", judo strategy is based on the idea of turning the opponent's strength into a disadvantage (Yoffie and Kwak 2002). In our model, as a construct, both the leader and the follower might not be able to open a store at the exact location of their choice. This gives both of them a reciprocal strength. Furthermore, the leader has lower fixed costs (another strength), but needs to open considerably more stores to prevent an entry. Since the follower cannot possibly sandwich the leader's stores due to his fixed cost disadvantage, he is not a big threat to the leader. Therefore, the leader finds it more beneficial to accommodate entry. Unlike in judo economics, however, here the follower gets at least half of the market. As the follower's fixed cost decreases he becomes a substantial threat to the leader, and hence, she chooses to prevent the entry.

Either by choice or out of necessity, once the leader lets the follower enter, her location strategy becomes quite insensitive to changes in her fixed cost and more so to changes in consumer density as seen in (19). The parameters, however, maintain the anticipated impact, i.e., the leader's stores become more dispersed in the areas where consumer density is low or her fixed cost is high. As a result of competition, the follower's fixed cost also has a slight impact on the leader's strategy. In the areas where the follower has high fixed cost, the leader's stores become less dispersed.

If the follower enters the market, his location strategy can be found by substituting (19) into (17) as:

$$g_1^*(x) = \left(\frac{4K_1(x)}{f(x)}\right)^{3/8} \left(\sqrt{2}K_0(x)\right)^{1/4},$$

and his profit is given as:

$$\Pi_0[g_1^*(x), g_0^*(x)] = \int_0^1 \left[f(x) - \left(\frac{2K_1(x)^5 f(x)^5}{K_0(x)^2}\right)^{1/8}\right] dx.$$

Similar to that of the leader, the follower's location strategy is also insensitive to small changes in consumer density and fixed costs. As consumer density decreases or his fixed cost increases, the follower's stores become more dispersed. However, as the leader's fixed cost increases, the follower's stores also become *more* dispersed. This is in contrast to the leader's strategy and may seem intriguing at first. However, as the leader's fixed cost increases she will open fewer stores giving the follower a chance to have at least as much market share with fewer stores.

Note that in order to keep the model tractable we have made a number of assumptions. For example, we have assumed that even though firms may have different fixed cost structures they have the same attraction for the consumers. However, a firm with higher fixed cost is usually characterized with larger stores and hence, wider merchandise assortment, or higher product availability, better infrastructure and hence, better service, or lower prices due to economies of scale. In those cases, duopoly regions would shift towards outer edges. We have also assumed that stores provide identical products or services. If there is some differentiation between the firms, duopoly regions become wider, reflecting a certain degree of imperfection in the competition.

5 Density Competition versus Location Competition

In this section we analyze the relationship between the model we have presented and a model of competition on the locations. Even though these two models are clearly not equivalent, our model descends directly from this location competition model by recognizing that firms might not be able to open stores at the exact locations of their choice. Therefore, we believe it is important to highlight the differences and similarities between these two models. In this section we will refer to our model as the *density competition* and to the other as the *location competition*.

Unfortunately, the location competition model is extremely hard to solve in general. Therefore, we simplify the problem by assuming uniform consumer density and constant fixed costs, i.e., $f(x) = 1$ and $K_i(x) = K_i$, for $i = 0, 1$ and for all $x \in [0, 1]$. Furthermore, we also ignore integrality in location competition as it has already been ignored in density competition. This assumption will only have a small effect on the results, especially when the potential number of stores is large.

Under uniform density and constant fixed costs, location competition is easy to solve. Suppose the follower finds it profitable to locate one store between two of the leader's stores. That store captures one half of the market in that interval, irrespective of where it locates. If this market justifies the opening fixed cost, then the follower can actually seize the whole market by opening two stores arbitrarily close to the leader's stores in the region and make twice the profit (this is a prelude to sandwiching). Therefore, the leader obtains either all or nothing between any of its two stores. This translates into

either the leader or the follower capturing all the market. In other words, either all of the leader's stores are sandwiched by the follower's stores and hence the leader does not even open a single store, or the leader prevents entry by opening a sufficient number of equidistant stores. Note that solutions where some stores are not equidistant are always dominated. Although the above description might suggest that sandwiching is a phenomenon only in the one-dimensional case, it is not so. An entrant also needs only two stores to capture all the demand of an incumbent store in the two-dimensional case. Let n_i^L be the number of stores firm $i = 0, 1$ opens and Π_i^L be the profit she or he makes under location competition. Similarly, let n_i^D and Π_i^D be those under density competition.

The leader can prevent entry if the follower's fixed cost advantage is not severe, i.e., $K_0 \leq 2K_1$. Under this condition the leader opens

$$n_0^L = 1/2K_1$$

stores, so that the follower does not find entry (strictly) profitable. Hence, the leader makes a profit of

$$\Pi_0^L = 1 - K_0/2K_1,$$

i.e., total revenue less total fixed cost. On the other hand if the leader's cost disadvantage is severe, i.e., $K_0 > 2K_1$, the follower then enters and captures the whole market and the leader exits. In theory, the follower can now capture the whole market with only one store. However, this outcome would not be stable because the leader may then re-enter. Therefore, the follower should open

$$n_1^L = 1/2K_0$$

stores to avoid the leader's re-entry, making a profit of

$$\Pi_1^L = 1 - K_1/2K_0.$$

Hence, the entry condition in location competition is simple: the follower enters if and only if $K_0 > 2K_1$. This is essentially condition (16). Therefore, if an entry occurs in location competition, it also occurs in density competition but not vice versa. As we have seen in the previous section, entry may occur in density competition even if this condition is not satisfied.

We have shown in the previous section that there are three possible market structures in density competition. Two of them, where the leader or the follower becomes a monopoly (labelled as MONOPOLY and UNPROFITABLE in Figure 8, respectively), are identical to the location competition in terms of the number of stores and the profits, which leads to the following proposition:

Proposition 1 *Under monopoly,* $n_i^D = n_i^L$, *and* $\Pi_i^D = \Pi_i^L$ *for* $i = 0, 1$.

We will therefore focus on the duopoly cases in the remainder of this section. As indicated earlier, there are two different cases where duopoly arises. First, the leader may have a severe cost disadvantage and yet stay in the market (region between conditions (16) and (21) in Figure 8), or she chooses to accommodate entry (regions between conditions (16) and (22) and conditions (20) and (22) in the same figure). We will refer to these cases as the *forced entry* and the *accommodated entry*, respectively. Note that an entry is accommodated only if $K_0 < 2K_1$, i.e., (16) holds and is forced only if $K_0 > 2K_1$.

Regardless of the type of entry, the following results can easily be obtained from Section 4 by simply setting $f(x) = 1$ and $K_i(x) = K_i$ in the corresponding formulas:

$$\begin{aligned}
n_0^D &= 1/g_0(x) = (K_1/8)^{1/4}(1/K_0)^{1/2}, \\
n_1^D &= 1/g_1(x) = (1/4K_1)^{3/8}(1/\sqrt{2}K_0)^{1/4}, \\
\Pi_0^D &= \left(\frac{K_1}{K_0}\right)^{1/4}\left[\frac{1}{\sqrt{2}}\left(\frac{K_1}{2}\right)^{3/8} - \left(\frac{K_0}{2}\right)^{3/4}\right], \\
\Pi_1^D &= 1 - \left(\frac{2K_1^5}{K_0^2}\right)^{1/8}.
\end{aligned}$$

When entry is accommodated, the leader opens fewer stores under density competition than under location competition. Otherwise, there would be no entry. Furthermore, she has to make more profit than she would if she prevented entry. Hence, the following proposition is almost immediate:

Proposition 2 *Under accommodated entry,*

a) $n_0^D \leq n_0^L$ *and* $\Pi_0^D \geq \Pi_0^L$,

b) $n_1^D \geq n_1^L = 0$ *and* $\Pi_1^D \geq \Pi_1^L = 0$,

c) $\frac{\Pi_0^D+\Pi_1^D}{\Pi_0^L+\Pi_1^L} \geq 1$,

d) $\frac{\Pi_0^D+\Pi_1^D}{\Pi_0^L+\Pi_1^L} \to \infty$ *as* $K_0 \to 2K_1$,

e) $\frac{n_0^D+n_1^D}{n_0^L+n_1^L} < 2.61$, *and*

f) $\frac{n_0^D+n_1^D}{n_0^L+n_1^L} \to 0$ *as* $K_0 \to 2K_1$ *and* $K_1 \to 0$.

Proof: Parts (a), (b), and (c) easily follow from the above discussion. For part (d), it is easy to show that, at the limit the denominator approaches to zero, whereas the numerator is strictly positive. For part (e), observe that

$$\begin{aligned}
\frac{n_0^D+n_1^D}{n_0^L+n_1^L} &= \frac{[(K_1/8)^{1/4}(1/K_0)^{1/2}]+[(1/4K_1)^{3/8}(1/\sqrt{2}K_0)^{1/4}]}{1/2K_1+0} \\
&= \left(\frac{2K_1^5}{K_0^2}\right)^{1/8}+\left(\frac{2K_1^5}{K_0^2}\right)^{1/4} \\
&\leq 4^{1/8}+4^{1/4} \\
&< 2.61,
\end{aligned}$$

where the first inequality comes from (20). As for part (f), first substitute $K_0 = 2K_1$ to obtain

$$\frac{n_0^D+n_1^D}{n_0^L+n_1^L} = [(K_1^3/2)^{1/8}+(K_1^3/2)^{1/4}] \to 0 \text{ as } K_1 \to 0.\square$$

Therefore, under location density competition the players in aggregate may open considerably more or fewer stores than under location competition. It is surprising to observe that even though they may open more stores they always make more profits. The reason is that these cases correspond to situations where the leader has higher fixed costs than the follower (for example, the region between conditions (16) and (22) in Figure 8). Hence, even though the leader opens fewer stores, the follower opens considerably more stores to offset and sometimes surpass the difference. Given above results it is fair to conclude that density competition is more relaxed than location competition because the profits in density competition are greater than or equal to those in location competition.

Under forced entry, the leader has a severe cost disadvantage and yet stays in the market. In location competition the leader exits the market and therefore ends up with zero profit. Hence, clearly the

leader makes a positive profit in density competition as compared to zero profit in location competition. We can make the following observations for this case:

Proposition 3 *Under forced entry,*

a) $n_0^D \geq n_0^L = 0$ *and* $\Pi_0^D \geq \Pi_0^L = 0$,

b) $n_1^D \leq n_1^L$,

c) $\frac{n_0^D + n_1^D}{n_0^L + n_1^L} < 1.82$, *and*

d) $\frac{n_0^D + n_1^D}{n_0^L + n_1^L} \to 0$ *as* $K_0 \to 2K_1$ *and* $K_1 \to 0$.

Proof: Part (a) again easily follows from the previous discussion. For part (b) it suffices to show that

$$\begin{aligned} \frac{n_1^D}{n_1^L} &= \frac{(1/4K_1)^{3/8}(1/\sqrt{2}K_0)^{1/4}}{1/2K_0} \\ &= 2^{1/8}(K_0^2/K_1)^{3/8} \\ &\leq 1, \end{aligned}$$

where the inequality comes from one of the conditions of forced entry, i.e., (21). We now prove parts (c) and (d). Note that

$$\begin{aligned} \frac{n_0^D + n_1^D}{n_0^L + n_1^L} &= \frac{(K_1/8)^{1/4}(1/K_0)^{1/2} + (1/4K_1)^{3/8}(1/\sqrt{2}K_0)^{1/4}}{0 + 1/2K_0} \\ &= (2K_0K_1)^{1/4} + (2K_0^2/K_1)^{3/8} \\ &\leq \left(2(1/2\sqrt[3]{2})(1/4\sqrt[3]{2})\right)^{1/4} + \left(2/\sqrt[3]{2}\right)^{3/8} \\ &< 1.82. \end{aligned}$$

The first part of the inequality comes from the fact that $K_0 \leq \tilde{K}_0 = 1/2\sqrt[3]{2}$ (see (23)) and from the fact that $2K_1 < K_0$, i.e., one of the conditions of forced entry. The second part of the inequality comes from the other condition of forced entry, i.e., (21).

As for part (d), first substitute $K_0 = 2K_1$ to obtain

$$\frac{n_0^D + n_1^D}{n_0^L + n_1^L} = [\sqrt{2K_1} + (8K_1)^{3/8}] \to 0 \text{ as } K_1 \to 0. \square$$

Under forced entry the results on profits are somewhat different than those observed under monopoly or accommodated entry. In fact we will show that profits under density competition may be less than those under location competition. Since the proof is somewhat longer, we will start with the following lemma:

Lemma 1 *Under forced entry, Π_0^D/Π_1^L and Π_1^D/Π_1^L are decreasing in K_0 for any K_1.*

Proof: Note that Π_0^D is decreasing and Π_1^L is increasing in K_0, hence, so is the first ratio. We now focus on the second ratio. Note that

$$\begin{aligned}\frac{\Pi_1^D}{\Pi_1^L} &= \frac{1-(2K_1^5/K_0^2)^{1/8}}{1-K_1/2K_0} \\ &= \frac{2K_0 - 2^{9/8}K_0^{3/4}K_1^{5/8}}{2K_0-K_1}.\end{aligned}$$

The denominator of the partial derivative

$$\frac{\partial \Pi_1^D/\Pi_1^L}{\partial K_0} = \frac{2^{1/8}K_1^{5/8}K_0^{3/4} - 2K_1 + 2^{9/8}K_1^{13/8}K_0^{-1/4}}{(2K_0-K_1)^2}$$

is always positive. Hence, it suffices to show that the numerator is negative under forced entry conditions. Note that

$$\begin{aligned}2^{1/8}K_1^{5/8}K_0^{3/4} - 2K_1 + 2^{9/8}K_1^{13/8}K_0^{-1/4} &\leq 2^{1/8}K_1^{5/8}(K_1/2^{1/3})^{3/8} - 2K_1 + 2^{9/8}K_1^{13/8}(2K_1)^{-1/4} \\ &= K_1 - 2K_1 + 2^{7/8}K_1^{11/8} \\ &= -K_1 + 2^{7/8}K_1^{11/8} \\ &\leq 0.\end{aligned}$$

In the first inequality, the first part comes from (21) and the second part from $K_0 \geq 2K_1$. The last inequality comes from $K_1 \leq \tilde{K}_0/2 = 1/4\sqrt[3]{2}$ (see (23)). □

We now present our main results on profits:

Proposition 4 *Under forced entry,*

a) $2/3 \leq \frac{\Pi_1^D}{\Pi_1^L} \leq 4/3$ *and*

b) $2/3 \leq \frac{\Pi_0^D + \Pi_1^D}{\Pi_0^L + \Pi_1^L} \leq 4/3$.

Proof: In part (a), since the ratio of the profits is decreasing in K_0 (due to above lemma), the bounds should be on the boundaries of the forced entry region, i.e., $K_0 = 2K_1$ and $\sqrt[3]{2}K_0^2 = K_1$. For $K_0 = 2K_1$,

$$\frac{\Pi_1^D}{\Pi_1^L} = \frac{1 - (K_1^3/2)^{1/8}}{3/4},$$

which is decreasing in K_1. Therefore, the lower and upper bounds on this boundary can be obtained by substituting $K_1 = 1/4\sqrt[3]{2}$ and $K_1 = 0$, respectively. These bounds are 2/3 and 4/3. Now consider the boundary $K_1 = \sqrt[3]{2}K_0^2$.

$$\frac{\Pi_1^D}{\Pi_1^L} = \frac{2^{2/3} - 2K_0}{2^{2/3} - K_0}$$

which is decreasing in K_0. The lower and upper bounds on this boundary can again be obtained by substituting $K_0 = 1/2\sqrt[3]{2}$ and $K_0 = 0$, respectively. Hence, we obtain the same points where the ratio approach to its bounds.

For part (b), note that this ratio is also decreasing in K_0 due to the lemma (also because $\Pi_0^L = 0$). The subsequent steps of the proof are virtually the same as in part (a). Observe that for $K_0 = 2K_1$

$$\frac{\Pi_0^D + \Pi_1^D}{\Pi_0^L + \Pi_1^L} = \frac{1 - (K_1/2^3)^{3/8} - (K_1/2)^{3/4}}{0 + 3/4},$$

which is also decreasing in K_1. Hence, the lower and upper bounds can be obtained by substituting $K_1 = 1/4\sqrt[3]{2}$ and $K_1 = 0$, respectively For $K_1 = \sqrt[3]{2}K_0^2$, note that $\Pi_0^D = 0$ because this is the boundary of the leader's profitability region. Therefore,

$$\frac{\Pi_0^D + \Pi_1^D}{\Pi_0^L + \Pi_1^L} = \frac{2^{2/3} - 2K_0}{2^{2/3} - K_0},$$

i.e., the same as in part (a). The bounds can be found similarly. Also note that at both of these points the leader makes zero profit under density competition as well as under location competition.□

It is interesting to observe that under forced entry, density competition may yield a lower total profit than location competition. Hence, for some values of fixed costs density competition is actually fiercer than location competition. This corresponds to the lower parts of the forced entry region, i.e., the regions where the leader's disadvantage is extreme yet she continues to survive. In fact this is the only

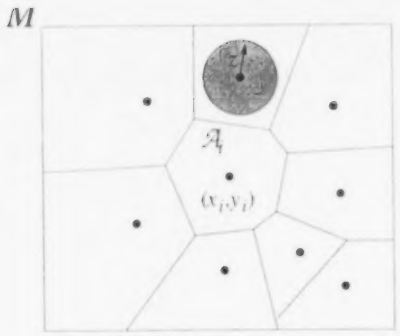

Figure 9: A market with branches over a two-dimensional region.

region in which such an outcome may arise. Everywhere else, the total profit under location density competition provides an upper bound (quite a poor one in some regions of accommodated entry) on the total profit under location competition.

Conclusions on the total number of stores are more diverse. In general, location density competition provides neither a lower bound nor an upper bound on the total number stores opened under location competition. Furthermore, there is no straightforward relationship between the total number of stores and the total profit. Finally, we have found that the total number of stores opened in density competition can be at most 2.61 times and 1.82 times that of the location competition under accommodated and forced entry, respectively.

6 Competition in a Two-Dimensional Market

We now extend the analysis to a two-dimensional market. We show that our methodology works just as well and all the results derived in the earlier sections remain valid under a mild condition. We provide the extension only for the follower's problem to demonstrate that the solutions in the one- and two-dimensional cases are identical. Although, we will carry out the analysis under the Euclidean distance assumption, the analysis can be adapted to other distance metrics. The analysis could also be extended to multiple dimensions easily, an issue not relevant to the store location but to product differentiation when products have multiple attributes.

Consider an arbitrary market $\mathcal{M}$ and a set of branches $(x_1, y_1), \ldots, (x_n, y_n)$ belonging to a firm. Assume that consumers are spread across the market according to a density $f(x, y)$. Given the branch locations and the distance metric, market areas $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of the branches can be characterized by an ordinary Voronoi diagram. An example configuration is depicted in Figure 9. However, whether a branch can capture all the consumers in its market area depends on the size and the shape of the area as well as the consumer utility. Similar to the earlier definition, let

$$u_i(x, y) = a - b\sqrt{(x - x_i)^2 + (y - y_i)^2}$$

be the utility the consumer located at (x, y) derives from patronizing branch (x_i, y_i). It is clear that a consumer located at (x, y) achieves the maximum utility in the market area $\mathcal{A}_i$ it belongs to because the branch of that market area is closest to that consumer. Let $z(x, y)$ denote the minimum distance from a consumer located at $(x, y) \in \mathcal{A}_i$ to a branch. The distribution of $z(x, y)$ can be approximated by:

$$P\{z(x, y) \leq z\} = \frac{\pi z^2}{A_i}, \text{ for } 0 \leq z \leq \sqrt{A_i/\pi},$$

where, A_i denote the area of $\mathcal{A}_i$. Note that the numerator is nothing but the area of a circle that has the branch location as the center and z as the radius. Hence, the probability is computed as the ratio of the area of the circle to the area of the market area (see Figure 9). As a word of caution, if another metric is used, then the area is no longer a circle. For example, under rectilinear metric the area will be a 45^o rotated square. We should note that this approximation is highly accurate if the market areas are close to circular shapes (or rotated squares for rectilinear metric). We now follow the same steps as in the one-dimensional case. First, we define a step function, $A_s(x, y)$, over the market as:

$$A_s(x, y) = A_i, \text{ for all } (x, y) \in \mathcal{A}_i.$$

Next, we approximate this step function with its continuous counterpart, $A(x, y)$. Figure 10 depicts the market areas and the corresponding step and smooth functions for an example configuration. Hence, the distribution of $z(x, y)$ can be approximated as:

$$P\{z(x, y) \leq z\} = \frac{\pi z^2}{A(x, y)}, \text{ for } 0 \leq z \leq \sqrt{A(x, y)/\pi}.$$

Note that $A(x, y)$ serves the same purpose as $g(x)$ in the one-dimensional case. Its role, however, is somewhat different. Here, $A(x, y)$ approximates the market area of each branch, whereas $g(x)$

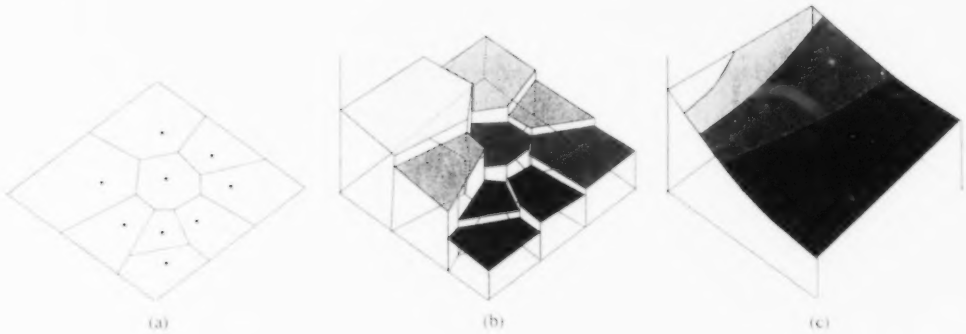

Figure 10: Market areas (a), the step function (b), and the smooth approximation (c) for an example.

approximates the distance between the branches. In each case, however, the firms express their location choices simply by these smooth functions. Note also that another difference between the one- and two-dimensional cases is the approximate distributions of the random variables that denote the minimum distance between a consumer and the stores (i.e., $z(x)$ and $z(x,y)$). In the one-dimensional case $z(x)$ has a uniform distribution whereas in the two-dimensional case $z(x,y)$ has a triangular distribution.

Let $u_j(x,y)$ be the maximum utility consumer (x,y) derives by patronizing one of the branches belonging to firm j and let $z_j(x,y)$ be the corresponding distance. Also let $A_j(x,y)$ denote the continuous function that approximates the market areas. We can compute the probability that the follower captures the consumer at (x,y) as:

$$\begin{aligned}
Q_1(x,y) &= P\{u_1(x,y) \geq u_0(x,y)\} \\
&= P\{z_1(x,y) \leq z_0(x,y)\} \\
&= \int_0^{\sqrt{A_0(x,y)/\pi}} P\{z_1(x,y) \leq z_0(x,y) | z_0(x,y) = z\} dF_{z_0(x,y)}(z) \\
&= \int_0^{\sqrt{A_0(x,y)/\pi}} P\{z_1(x,y) \leq z\} \frac{2\pi z}{A_0(x,y)} dz \\
&= \begin{cases} \frac{A_0(x,y)}{2A_1(x,y)} & \text{if } A_1(x,y) \geq A_0(x,y) \\ 1 - \frac{A_1(x,y)}{2A_0(x,y)} & \text{if } A_1(x,y) \leq A_0(x,y). \end{cases}
\end{aligned}$$

The treatment of fixed costs is essentially the same as in the one-dimensional case. Let $K_1(x,y)$ be the fixed cost of opening a branch at location (x,y), which is also assumed to be a slowly varying

function of x and y. We repeat the earlier process as follows (we drop the index 1 only in this part for the sake of clarity):

$$\begin{aligned}
\sum_i K(x_i, y_i) &= \sum_i K(x_i, y_i) \int_{\mathcal{A}_i} \frac{1}{A_i} dxdy \\
&= \sum_i K(x_i, y_i) \int_{\mathcal{A}_i} \frac{1}{A_s(x,y)} dxdy \\
&\approx \sum_i \int_{\mathcal{A}_i} \frac{K(x,y)}{A_s(x,y)} dxdy \text{ (because } K(x,y) \text{ is slowly varying)} \\
&= \int_{\mathcal{M}} \frac{K(x,y)}{A_s(x,y)} dxdy \\
&\approx \int_{\mathcal{M}} \frac{K(x,y)}{A(x,y)} dxdy.
\end{aligned}$$

Hence, a profit-maximizing follower solves the following problem:

$$\max_{A_1(x,y) \geq 0} \Pi_1[A_1(x,y), A_0(x,y)] = \int_{\mathcal{M}} \left[Q_1(x,y) f(x,y) - \frac{K_1(x,y)}{A_1(x,y)} \right] dxdy. \tag{24}$$

Note that (24) is essentially the same as (13). Hence, we skip the rest of the analysis, as it would be a replication of the earlier sections. Given our treatment it is fair to conclude that earlier results generalize to two-dimensional markets under a mild condition, i.e., market areas have shapes closer to their ideals under given distance metrics.

7 Concluding Remarks

We have discussed the problem of determining the density of retail branches under competition. Our analysis has produced several insightful and intuitive results without sacrificing much of the information a decision maker might need. We have focused on the investigation of the impact of fixed costs on location patterns. We should note, however, that the models allow many other parametric analysis. Furthermore, our results are general in one- and two-dimensional markets. We have observed that the leader has the first mover advantage, which she uses either to prevent or to accommodate entry. Furthermore, we have shown that the leader can make positive profits even if she is severely cost disadvantaged. For example, in a typical location competition game a leader under such intense cost disadvantage would have all of her stores sandwiched between rival stores. A similar advantage is

also available to the follower, albeit at a much reduced level. Therefore, it is fair to conclude that in cases where firms make location density decisions, the competition is generally more relaxed, because a certain level of indeterminacy in the exact locations acts as if the firms are somewhat differentiated.

We have also established that the primary effect of consumer density and fixed cost is on the market structure. It has been shown that the firms' entry and entry deterrence decisions are quite sensitive to changes in these parameters, but not their location strategies once both are present in the market. The secondary role of consumer density and fixed costs is on the firms' location strategies once the market structure is defined. We have observed that these parameters have a small impact on the firms' strategies except for a leader who vigorously tries to maintain her monopoly position. However, we should note that under elastic demand conditions, these parameters would have a higher impact.

Many of the assumptions we have made throughout the paper could be relaxed and handled by our models, leading to future research avenues. First, elastic demand, more elaborate choice processes, random utilities, and consumer search patterns could be incorporated. Second, we have assumed common ownership of a firm as well as its branches. However, there are many cases where different ownerships and a variety of franchising contracts exist. Thus, including contracts such as those governing the exclusive territories would be an important contribution. Third, economies of scale observed in the areas of regional management, distribution, advertisement, and work force training could also be incorporated. Finally, our models could be extended to include such store characteristics as service quality, merchandise assortment, and price.

Appendix

Here we prove that the integrand in the objective function (18) is quasi-concave. Let us write the integrand as

$$\pi(w) = \frac{A}{\sqrt{w}} - \frac{B}{w},$$

where, $w = g_0(x)$, $A = (K_1(x)/2f(x))^{1/2}$, and $B = K_0(x)$. Given w_1 and w_2, and $\lambda \in (0,1)$ we will prove that $\pi(w_0) \geq \min\{\pi(w_1), \pi(w_2)\}$, where, $w_0 = (1-\lambda)w_1 + \lambda w_2$. Rewrite the integrand as:

$$\pi(w) = \frac{A\sqrt{w} - B}{w}.$$

It is clear that the numerator is strictly concave. Thus

$$\begin{aligned}
\pi(w_0) &= \frac{A\sqrt{w_0} - B}{w_0} \\
&= \frac{A\sqrt{w_0} - B}{(1-\lambda)w_1 + \lambda w_2} \\
&\geq \frac{(1-\lambda)(A\sqrt{w_1} - B) + \lambda(A\sqrt{w_2} - B)}{(1-\lambda)w_1 + \lambda w_2} \\
&= \frac{(1-\lambda)w_1\pi(w_1) + \lambda w_2\pi(w_2)}{(1-\lambda)w_1 + \lambda w_2} \\
&= \frac{(1-\lambda)w_1}{(1-\lambda)w_1 + \lambda w_2}\pi(w_1) + \frac{\lambda w_2}{(1-\lambda)w_1 + \lambda w_2}\pi(w_2) \\
&\geq \min\{\pi(w_1), \pi(w_2)\}.\square
\end{aligned}$$

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